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Some properties of the Itô–Wiener expansion of the solution of a stochastic differential equation and local times

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Abstract

In this paper, we use the formula for the Itô–Wiener expansion of the solution of the stochastic differential equation proven by Krylov and Veretennikov to obtain several results concerning some properties of this expansion. Our main goal is to study the Itô–Wiener expansion of the local time at the fixed point for the solution of the stochastic differential equation in the multidimensional case (when standard local time does not exist even for Brownian motion). We show that under some conditions the renormalized local time exists in the functional space defined by the L_2-norm of the action of some smoothing operator.

MSC: 60J55; 60H10

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1. Introduction

Local time of a random process is usually defined as the density of the occupation measure of the process with respect to Lebesgue measure, or, alternatively, as an additive functional for Markov processes. But there exists another definition: local time can be defined as a limit of approximations. This definition allows an interesting modification, which was discovered to be useful for the self-intersection local time for two-dimensional Brownian motion in L_2

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(see for example [11]). The limit in L_2 of the approximations does not exist in this case. But, if we change these approximations by subtracting their mathematical expectation, the modified approximations will converge in L_2. This limit is called the renormalized local time. It plays a significant role in the description of self-intersections for the two-dimensional Brownian motion. For example, it is known that the asymptotics of the area of the Wiener sausage contains renormalized local time [7].

In recent years a new approach has been developed that utilizes Gaussian structure of the process. The general idea is as follows: study Itô–Wiener expansion of the approximations for local time and find some properties of the expansion members that allow us to study their convergence. This convergence can be used to obtain the existence of local time defined as a limit of approximations in some functional space (for example in Watanabe–Sobolev space). The concept of renormalization appears naturally in this approach, since the first member of Itô–Wiener expansion is the mathematical expectation. It means that we can study renormalized local time (understanding renormalization as the subtraction of the mathematical expectation) using this approach with minimal additional effort. This idea was used successfully in many papers. Imkeller, Perez-Abreu, Vives [5] proved that it is possible to define the renormalized self-intersection local time for d-dimensional Brownian motion as an element of Sobolev space, but only for the case $d = 2$. Several authors considered different Gaussian processes, with many results produced for fractional Brownian motion (see [9,2,4]). The general result for a class of Gaussian processes was presented in our earlier work [12], along with the application to the self-intersection local time for fractional Brownian motion.

It is well-known that classical local time (at the fixed point) does not exist for Brownian motion in the dimensions greater than one, and the same is true, generally speaking, for diffusion processes. However we can still define a "generalized version" of local time. Dorogovtsev and Bakun [1] found that it is possible to define a generalized local time for multidimensional Brownian motion. This generalized local time is a limit of the renormalized approximations in some functional space, where renormalization is the subtraction of several members of the Itô–Wiener expansion. The functional space has to include the additional "smoothing" for the kernels of the Itô–Wiener expansion (otherwise their standard L_2 norm grows to infinity). Originally the idea, that was proposed by Dorogovtsev, was to extend the well-known renormalization result for the self-intersection local time for two-dimensional Brownian motion to another situation.

In this paper we chose to study a more general case in the similar fashion. We have Brownian motion replaced by the solution of the stochastic differential equation (SDE). We use the known approach outlined above, and study the Itô–Wiener expansion of the local time for the solution of SDE, even though the local time in the usual meaning may not exist. The functional space used in our paper (and in [1]) is relatively simple and may be considered natural, since it is connected to the well-known second quantization operator $\Gamma(A)$. The choice of this norm is justified by the fact that it provides an appropriate "smoothing" for the kernels of the Itô–Wiener expansion. Our results may also be seen as some properties of the Itô–Wiener expansion for the solution of SDE, without mentioning local times.

Our interest in this investigation can be explained more fully by the following. Local time is a fundamental notion for random processes: if we know something about local time then we have better understanding of the behaviour of the process. Similarly if we know something about the Itô–Wiener expansion of a random variable then we can find out more about this random variable. In our case there may be no actual local time as a random variable but the connection between the Itô–Wiener expansion and the geometry of the solution of SDE may still exist. Therefore we hope that the existence of a generalized local time can provide more information about the process. In the future our results may be extended and possibly used to study the existence and uniqueness of the solution or to describe self-intersections for the solution of SDE. Also it is interesting to see how the properties of the process change when we replace Brownian motion by the solution of SDE. This replacement can be viewed as the action of so-called Itô map. The properties of the Itô–Wiener expansion that we study may be related to some properties of Itô map.

The main result of this paper is the extension of the result about the existence of generalized local time from [1] to the solution of SDE. We assume the continuity of the coefficients of SDE, the condition of uniform strict ellipticity and additionally that these coefficients are smooth and bounded together with all their derivatives in the space variable. Under these conditions it turns out to be possible to investigate the behaviour of the Itô–Wiener expansion of a function of the solution. In particular we prove some estimates for the kernels of the Itô–Wiener expansion, similar to those existing for the transition density of the process. Then we consider the functional space similar to the one used in [1] and find a representation of the norm of the local time approximations, that allows us to prove the existence of the renormalized local time as an element of this functional space. We use two classical results: the representation of the Itô–Wiener expansion of the solution of SDE (we call it Krylov–Veretennikov representation), proven by Veretennikov and Krylov in [14] and Gaussian estimates for the derivatives of the fundamental solution of parabolic PDE [3]. The second result is useful since Krylov–Veretennikov representation can be written using the fundamental solution of parabolic PDE associated with SDE. Note that we consider the local time at a single point for d-dimensional diffusion process with $d \geqslant 2$ (case $d = 1$ is formally included but trivial), which does not exists as a random variable even with renormalization, therefore all known results for classical local times of diffusion processes are not applicable.

In the next section we introduce some basic definitions and notation. After that we use two known results: Krylov–Veretennikov representation and the existence of the fundamental solution of parabolic PDE to find the Itô–Wiener expansion of the local time approximations. The Section 4 is fully devoted to the properties of the fundamental solution of parabolic PDE. We extend some classical Gaussian estimates in the case when the coefficients of PDE have uniformly bounded derivatives of any order. These estimates are applicable to the representation of the Itô–Wiener expansion of the local time approximations. In the Section 5 we show some interesting examples when the coefficients of SDE are not smooth. We show that non-smoothness leads to the violation of Gaussian estimates for Itô–Wiener expansion. In the last section we prove our main result about the existence of the renormalized local time for the solution of SDE.

2. Definitions and notation

Consider a Cauchy problem for stochastic differential equation

$$dX_t = a(t, X_t)dt + \sigma(t, X_t)dW_t, X_s = x \tag{1}$$

where $X_t \in \mathbb{R}^d$, a is a measurable bounded d-vector function, σ is a measurable bounded $d \times d$-matrix function and W is a standard Wiener process in $\mathbb{R}^d$. Denote $b = \sigma\sigma^T$. Suppose that b is strictly uniformly positive definite, i.e.

$$\exists \delta > 0, \forall t \in \mathbb{R}, x \in \mathbb{R}^d, y \in \mathbb{R}^d : (b(t, x)y, y) > \delta \|y\|^2. \tag{2}$$

It is known that under given conditions this stochastic equation has a weak solution (see [6]).

Denote, for any $f \in C^2(\mathbb{R}^d)$,

$$S_k^{t,x} f(x) = \sum_{i=1}^{d} \sigma_{ik}(t,x) \frac{\partial f}{\partial x_i}(x) \tag{3}$$

$$L^{t,x} f(x) = \sum_{i=1}^{d} a_i(t,x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{k=1}^{d} (\sigma_{ik}\sigma_{jk})(t,x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x). \tag{4}$$

Let $T_{s,t}$, $s < t$ be a set of operators that define the solution of the following Cauchy problem (which is solved backwards in time):

$$\frac{\partial}{\partial s} u(s,x) + L^{s,x} u(s,x) = 0, \quad s < t;\ u(t,x) = f(x) \tag{5}$$

and by the definition $T_{s,t} f(x) = u(s,x)$. It is well-known that $T_{s,t} f(x) = Ef(X_t)$, where X_t is a solution of (1).

Denote as $C_b^\infty(\mathbb{R}^d)$ a space of infinitely differentiable functions on $\mathbb{R}^d$ that are bounded together with all their derivatives. We say that family of functions $\{f_u, u \in U\}$ belong uniformly to $C_b^\infty(\mathbb{R}^d)$ with constants $M_q, q = (q_1, \ldots, q_n), q_i = 1, \ldots, d$ if for all $n, q, u \in U$ and $x \in \mathbb{R}^d$:

$$\left| \frac{\partial}{\partial x_{q_1}} \cdots \frac{\partial}{\partial x_{q_n}} f_u(x) \right| \leqslant M_q. \tag{6}$$

We are going to investigate properties of X_t using theory of Gaussian spaces and Malliavin calculus. Therefore we need some basic facts, that can be found for example in [15,8]. Let $W_t \in \mathbb{R}^d, t \in [0,1]$- be a d-dimensional Wiener process (same as before except that we restrict it to the time interval $[0,1]$), $\mathfrak{F}^W - \sigma$-algebra generated by W. Denote $\Delta_n[0,1] = \{(t_1, \ldots, t_n) \in \mathbb{R}^n : 0 < t_1 < \cdots < t_n < 1\}$. Suppose that random variable ξ is $\mathfrak{F}^W$-measurable and lies in $L_2(\Omega)$, then there exists a unique sequence of functions $a_m \in L_2(\Delta_n[0,1])$ (we may suppose that $a_m \in L_2([0,1]^n)$ by setting a_m equal to zero where it is not defined) indexed by n and multiindex $m = (m_1, \ldots, m_n), m_i = 1, \ldots, d$, such that:

$$\xi = \sum_{n=0}^{\infty} \sum_{m=(1,\ldots,1)}^{(d,\ldots,d)} \int_{\Delta_n[0,1]} a_m(t_1, \ldots, t_n) dW_{m_1}(t_1) \ldots dW_{m_n}(t_n).$$

This sum consists of orthogonal elements of $L_2(\Omega)$ and defines decomposition of $L_2(\Omega)$ into a direct sum of subspaces. It is called Itô–Wiener expansion or sometimes (when dealing with the general case of Gaussian spaces) chaos decomposition. On the linear space of random variables from $L_2(\Omega)$, such that the sum above is finite, we can define the following norm, for all $\alpha \in \mathbb{R}$:

$$\|\xi\|_{2,\alpha}^2 = \sum_{n=0}^{\infty} (1+n)^\alpha \sum_{m=(1,\ldots,1)}^{(d,\ldots,d)} \|a_m\|_2^2$$

where $\|a_m\|_2$ is the norm of a_m in $L_2([0,1]^n)$. The completion of this space is denoted $D_{2,\alpha}$ and called Sobolev space (or Watanabe–Sobolev space). Similarly we can define spaces $\Phi_2(A)$ using norm

$$\|\xi\|_{2,A}^2 = \sum_{n=0}^{\infty} \sum_{m=(1,\ldots,1)}^{(d,\ldots,d)} \|(A^{\otimes n} a)_m\|_2^2 \tag{7}$$

where A is a "smoothing" linear operator on the space $(L_2([0,1]))^d$ and $A^{\otimes n}$ acts on the $\{a_m, m = (m_1, \dots, m_n)\}$ as an element of $(L_2([0,1])^d)^{\otimes n}$. We will define this action precisely below and only for a certain type of operators, to avoid unnecessary complications.

This norm can be written as

$$\|\xi\|^2_{2,A} = |\Gamma(A)\xi|^2_{2,0} = E(\Gamma(A)\xi)^2 \tag{8}$$

where $\Gamma(A)$ is so-called "second quantization operator", which can be defined by (see [13])

$$\Gamma(A)\xi = \sum_{n=0}^{\infty} \sum_{m=(1,\dots,1)}^{(d,\dots,d)} \int_{\Delta_n[0,1]} (A^{\otimes n} a)_m(t_1, \dots, t_n) dW_{m_1}(t_1) \dots dW_{m_n}(t_n).$$

in the case when $\|A\| \leqslant 1$.

In this paper we consider only multiplication operators of form:

$$(A\varphi)_i(t) = \psi_i(t)\varphi_i(t), \varphi \in (L_2([0,1]))^d \tag{9}$$

where ψ_i are measurable functions such that:

$$\sup_{i,t} |\psi_i(t)| \leqslant \beta < 1. \tag{10}$$

Now $A^{\otimes n}$ acts on the $\{a_m, m = (m_1, \dots, m_n)\}$ in the following way

$$(A^{\otimes n} a)_m(t_1, \dots, t_n) = \psi_{m_1}(t_1) \dots \psi_{m_n}(t_n) a_m(t_1, \dots, t_n)$$

There is an alternative formula for the action of the operator $\Gamma(A)$ with such A:

$$\Gamma(A)\xi = \tilde{E}(\xi(\hat{W})) \tag{11}$$

where $\hat{W}$ is a d-dimensional Wiener process, defined by

$$d\hat{W}_i(t) = \psi_i(t)dW_i(t) + \sqrt{1-\psi_i^2(t)}d\tilde{W}_i(t).$$

$\tilde{W}$ is a d-dimensional Wiener process independent of W and $\tilde{E}$ is the mathematical expectation over probability space of $\tilde{W}$. This formula is easy to get from the definition of $\Gamma(A)$. We also use the following notation:

$$(\xi_1, \xi_2)_{2,A} = E(\Gamma(A)\xi_1 \Gamma(A)\xi_2)$$

for ξ_1, ξ_2 that are $\mathfrak{F}^W$-measurable and in $L_2(\Omega)$.

In this article we define local time using a family of approximations as follows. Denote

$$I_{LT}(f) = \int_0^1 f(X_t)dt$$

where X_t is a solution of (1) and let

$$f_\varepsilon(x) = (2\pi\varepsilon)^{-d/2} e^{-\frac{\|x\|^2}{2\varepsilon}}$$

be a family of functions approximating δ_0 as $\varepsilon \to 0+$.

Definition 1. If the limit

$$\lim_{\varepsilon \to 0+} I_{LT}(f_\varepsilon(\cdot - x))$$

exists in $D_{2,\alpha}$ we call it the local time of the process X_t at x in $D_{2,\alpha}$. If it exists in $\Phi_2(A)$ we call it the local time of the process X_t at x in $\Phi_2(A)$.

Denote

$$I^n_{LT}(f) = I_{LT}(f) - \sum_{k=0}^{n} \sum_{|m|=k} \xi_m$$

where

$$\xi_m = \int_{\Delta_n[0,1]} a_m(t_1, \dots, t_n)\, dW_{m_1}(t_1) \dots dW_{m_n}(t_n)$$

is an element of Itô–Wiener expansion of $I_{LT}(f)$ with multiindex m.

Definition 2. If the limit

$$\lim_{\varepsilon \to 0+} I^n_{LT}(f_\varepsilon(\cdot - x))$$

exists in $D_{2,\alpha}$ we call it n-renormalized local time of the process X_t at x in $D_{2,\alpha}$. If it exists in $\Phi_2(A)$ we call it n-renormalized local time in $\Phi_2(A)$ of the process X_t at x.

In this article we do not use the definition with convergence in $D_{2,\alpha}$ (since it does not work for our case), but this definition is widely used in several papers including our previous works (see [12]). Therefore it is useful to discuss its relation to the convergence in $\Phi_2(A)$. First of all if $\|A\| < 1$ (we always suppose this) then it is easy to see that $\Phi_2(A)$ norm is dominated by any $D_{2,\alpha}$ norm. Moreover if A is a multiplication operator (again we use only this kind of operators) then we get an additional "smoothing" effect on the kernels of the Itô–Wiener expansion with a suitable choice of multiplier. In fact this "smoothing" effect is the reason for us to introduce $\Phi_2(A)$ norm. We need it since the kernels of Itô–Wiener expansion of local time may have singularities and may not belong to $L_2([0,1]^n)$ (it happens even for standard Wiener process in the dimension $d \geqslant 2$, see [1] and example below). This is why our first task is to study these singularities. We are going to achieve that with some suitable representation of the Itô–Wiener expansion of $I_{LT}(f)$.

3. Itô–Wiener expansion of solutions of SDE

The purpose of this section is to find a representation of the Itô–Wiener expansion of local time for X_t, which is a solution of (1). We will rewrite this representation using the fundamental solution G of associated PDE (5), i.e. function $G(s, t, x, y)$ defined for $s < t$ and $x, y \in \mathbb{R}^d$, such that:

1. G is jointly continuous in all variables for $s < t$ and $x, y \in \mathbb{R}^d$.
2. G is one time differentiable with respect to s and two times differentiable with respect to x for all $s < t$ and $x, y \in \mathbb{R}^d$ and satisfies $\frac{\partial}{\partial s} G(s, t, x, y) + L^{s,x} G(s, t, x, y) = 0$.
3. For any bounded continuous function f on $\mathbb{R}^d$:

   $$\lim_{s \to t-} \int_{\mathbb{R}^d} G(s, t, x, y) f(y) dy = f(x)$$

   and the limit exists uniformly on the compact sets.

We denote as $\{a_m(s, t_1, \ldots, t_n, t, x) : m = (m_1, \ldots, m_n), m_i = 1, \ldots, d\}$ the kernels of the Itô–Wiener expansion of $E(f(X_t)/\mathfrak{F}_t^W)$, where $0 < s < t_1 < \cdots < t_n < t < 1$ (s were fixed as a starting moment of time for our Cauchy problem, now we additionally assume that $s \in (0, 1)$), f – a bounded measurable function and (X, W) is a weak solution of (1). Denote as $\{b_m(s, t_1, \ldots, t_n, t, x) : m = (m_1, \ldots, m_n), m_i = 1, \ldots, d\}$ the kernels of the Itô–Wiener expansion of $\int_s^t E(f(X_r)/\mathfrak{F}_r^W)dr$. Note that

$$\int_s^t E(f(X_r)/\mathfrak{F}_r^W)dr = E\left(\int_s^t f(X_r)dr/\mathfrak{F}_t^W\right) \quad \text{a.s.}$$

due to the well-know fact that $E(f(X_r)/\mathfrak{F}_r^W) = E(f(X_r)/\mathfrak{F}_t^W)$ a.s. if $r < t$ (it follows from the definition of weak solution).

Theorem 1. *Suppose that b satisfies (2), a and b are both continuous and bounded above by constant M_2 and satisfy Holder condition with respect to x with constant M_1 and exponent $\theta \leqslant 1$ for all t, b is uniformly continuous on $[0, 1] \times \mathbb{R}^d$ and σ is a measurable function. Then for any weak solution (X, W) of (1):*

- *The kernels of Itô–Wiener expansion of $E(f(X_t)/\mathfrak{F}_t^W)$ can be represented as:*

$$a_m(s, t_1, \ldots, t_n, t, x) = \int_{\mathbb{R}^d} q_m(s, t_1, \ldots, t_n, t, x, y) f(y)dy \tag{12}$$

where

$$q_m(s, t_1, \ldots, t_n, t, x, y) = \int_{\mathbb{R}^{dn}} G(s, t_1, x, z_1) S_{m_1}^{t_1, z_1} G(t_1, t_2, z_1, z_2) \\ \ldots S_{m_{n-1}}^{t_{n-1}, z_{n-1}} G(t_{n-1}, t_n, z_{n-1}, z_n) S_{m_n}^{t_n, z_n} G(t_n, t, z_n, y) dz_1 \ldots dz_n \tag{13}$$

and each $S_i^{u,z}$ is operator defined by (3) which acts on variable z.

- *Each q_m satisfies the following inequality for $0 < s < t < 1$:*

$$|q_m(s, t_1, \ldots, t_n, t, x, y)| \leqslant C_n((t_2 - t_1) \ldots (t - t_n))^{-1/2}(t - s)^{-d/2} \exp^{-\gamma \frac{\|x-y\|^2}{2(t-s)}} \tag{14}$$

where $C_n, n = 0, 1, \ldots$ and γ are some positive constants depending only on a and b.

- *The kernels of Itô–Wiener expansion of $\int_s^t E(f(X_r)/\mathfrak{F}_r^W)dr$ can be represented as:*

$$b_m(s, t_1, \ldots, t_n, t, x) = \int_{\mathbb{R}^d} r_m(s, t_1, \ldots, t_n, t, x, y) f(y)dy \tag{15}$$

where

$$r_m(s, t_1, \ldots, t_n, t, x, y) = \int_{t_n}^t q_m(s, t_1, \ldots, t_n, r, x, y)dr. \tag{16}$$

We are going to prove this theorem using two known theorems from theory of SDE and PDE. Our starting point is a well-known result proved by Veretennikov and Krylov in [14] (first appeared in a joint work of Krylov and Zvonkin [16]) that gives an explicit formula for the Itô–Wiener expansion of some function of X_t.

Theorem 2 (*[14]*). *If σ, a are bounded measurable, $b = \sigma\sigma^T$ is uniformly continuous on $[0, 1] \times \mathbb{R}^d$ and satisfies (2), then kernels a_m can be represented as*

$$a_m(s, t_1, \ldots, t_n, t, x) = T_{s,t_1} S_{m_1}^{t_1, \cdot} T_{t_1, t_2} S_{m_2}^{t_2, \cdot} \ldots S_{m_n}^{t_n, \cdot} T_{t_n, t} f(x). \tag{17}$$

The condition of uniform continuity for b is a stronger version of an additional restriction in the original result by Krylov and Veretennikov (as authors suggested themselves). Later we will show that we can drop this condition for smooth a and b.

Theorem 2 was proved in [14] using repeated application of Itô formula. Another way to prove this representation is to use second quantization operator $\Gamma(A)$ (or Ornstein–Uhlenbeck semigroup, i.e take $A = e^{-\lambda} I$). We will give more information about this approach later, since we want to use it directly to estimate $\Phi_2(A)$-norm of the renormalized local time rather than through the representation from Theorem 2.

Theorem 2 can be used to find the Itô–Wiener expansion of $\int_0^1 f(X_t)dt$.

Theorem 3. *Under conditions of Theorem 2 the kernels*

$$\{b_m(s, t_1, \dots, t_n, t, x) : m = (m_1, \dots, m_n), m_i = 1, \dots, d\}$$

of the Itô–Wiener expansion of $\int_s^t E(f(X_r)/\mathfrak{F}_r^W)dr$ can be represented as

$$b_m(s, t_1, \dots, t_n, t, x) = \int_{t_n}^t a_m(s, t_1, \dots, t_n, r, x)dr \tag{18}$$

where $a_m(s, t_1, \dots, t_n, r, x)$ are the kernels of the Itô–Wiener expansion of $E(f(X_r)/\mathfrak{F}_r^W)$ as in formula (17) from Theorem 2.

Proof. The representation of a_m in Theorem 2 provides its joint measurability. Therefore we only need to prove that the integral with respect to dr can be put inside of the sum in the Itô–Wiener expansion and inside of the multiple stochastic integral. The first is a consequence of the following obvious fact: $\int_0^1 \sqrt{Ef^2(X_t)}dt < +\infty$, as shown in [12] in the proof of Theorem 3.1 and the second is a well-known property of stochastic integral, which holds under the same condition. The idea here is to estimate the integral using the norm in $L_2(\Omega)$. For example $a_m(s, t_1, \dots, t_n, r, x)$ is integrable with respect to dr (absolutely in Lebesgue sense) for almost all $t_1, \dots, t_n$ since

$$\begin{aligned}
&\left\| \int_s^t |\mathbf{1}_{\{t_n < r\}} a_m(s, t_1, \dots, t_n, r, x)| dr \right\|_2 \\
&\quad \leqslant \int_s^t \|\mathbf{1}_{\{t_n < r\}} a_m(s, t_1, \dots, t_n, r, x)\|_2 dr \\
&\quad \leqslant \int_s^t \|f(X_r)\|_{2,0} dr = \int_0^1 \sqrt{Ef^2(X_t)}dt < +\infty
\end{aligned}$$

where $\|\cdot\|_2$ is the norm in $L_2(\Delta_n[0, 1])$ of a function of $t_1, \dots, t_n$ and $\|\cdot\|_{2,0}$ is the norm in $L_2(\Omega) = D_{2,0}$. □

Our next step is to recall the well-known result from the theory of parabolic partial differential equations (see for example [10]).

Theorem 4. *Suppose that a and b are continuous and bounded above by constant M_2, satisfy Holder condition with respect to x with constant M_1 and exponent $\theta \leqslant 1$ for all t and b satisfies (2). Then there exists unique fundamental solution G for differential operator $\frac{\partial}{\partial t} + L^{t,\cdot}$. Additionally G is positive and satisfies the following inequalities:*

$$G(s, t, x, y) \leqslant C(t - s)^{-d/2} \exp^{-\gamma \frac{\|x-y\|^2}{2(t-s)}}$$

$$\left|\frac{\partial}{\partial x_i} G(s,t,x,y)\right| \leqslant C(t-s)^{-(d+1)/2} \exp^{-\gamma \frac{\|x-y\|^2}{2(t-s)}}$$
$$\left|\frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} G(s,t,x,y)\right| \leqslant C(t-s)^{-(d+2)/2} \exp^{-\gamma \frac{\|x-y\|^2}{2(t-s)}}$$

with some positive constants C and γ depending only on M_1, θ, M_2, δ and d.

As a consequence of this theorem we can represent a solution of Cauchy problem (5) for a bounded continuous functions f using fundamental solution.

$$u(s,x) = \int_{\mathbb{R}^d} G(s,t,x,y) f(y) dy$$

It is well-known that this solution is unique in the class of solutions growing not faster than $e^{\beta \|x\|^2}$ if the coefficients a and b are bounded. The fundamental solution G is also a kernel for $T_{s,t}$.

Using the fundamental solution we can finally rewrite the representations (17) and (18) as a repeated convolutions of the fundamental solution with itself (under the action of some first order differential operator).

Proof of Theorem 1. First of all due to Theorem 4 the fundamental solution G exists. This solution and its first derivative with respect to x are bounded by Gaussian densities. We use the estimate for derivative of fundamental solution as follows:

$$|S_k^{s,x} G(s,t,x,y)| \leqslant d \sup_{i,x,r} |\sigma_{ik}(r,x)| C(t-s)^{-(d+1)/2} \exp^{-\gamma \frac{\|x-y\|^2}{2(t-s)}}.$$

Since

$$\sup_{i,t,x} |\sigma_{ik}(t,x)| \leqslant \sup_{i,t,x} \sqrt{\sum_{k=1}^{d} (\sigma_{ik}(t,x))^2} = \sup_{i,t,x} \sqrt{b_{ii}(t,x)} \leqslant \sqrt{M_2}$$

we obtain:

$$\begin{aligned}
|q_m(s,t_1,\ldots,t_n,t,x,y)| \leqslant\ & (dM_2)^n C^{n+1} \int_{\mathbb{R}^{dn}} (t_1-s)^{-d/2} \exp^{-\gamma \frac{\|x-z_1\|^2}{2(t_1-s)}} \\
& \times (t_2-t_1)^{-(d+1)/2} \exp^{-\gamma \frac{\|z_2-z_1\|^2}{2(t_2-t_1)}} \ldots \\
& \times (t-t_n)^{-(d+1)/2} \exp^{-\gamma \frac{\|y-z_n\|^2}{2(t-t_n)}} dz_1 \ldots dz_n \\
=\ & (dM_2)^n C^{n+1} (2\pi)^{nd/2} ((t_2-t_1)\ldots(t-t_n))^{-1/2} \\
& \times (t-s)^{-d/2} \exp^{-\gamma \frac{\|x-y\|^2}{2(t-s)}}.
\end{aligned}$$

Since assumptions on a and b in Theorem 1 are stronger than in Theorem 2 we can apply Theorem 2. It means that we have the representation (17) for a_m. We can replace the action of semigroup $T_{s,t}$ in it by the convolution using kernel G. We may differentiate with respect to x in the action of $S^{s,x}$ under all integrals. The resulting multiple integral exists and can be taken in any order due to the Gaussian estimate on $|S_k^{s,x} G(s,t,x,y)|$ that we already mentioned. This immediately means that (12) and (13) are proved. To prove (15) and (16) we need to show that integrating q_m first with respect to dt then with respect to $f(x)dx$ is the same as vice versa. The difficulty here is that q_m may have singularity near $t = t_n$ (see examples below). But as we can see the estimate (14) shows that this singularity is not worse than $(t - t_n)^{-1/2}$, so $|q_m|$ is integrable with respect to dt and the theorem is proved. □

In this section we established that if we want to estimate the kernels of the Itô–Wiener expansion of the local time for a solution of SDE we can work with the fundamental solution of associated PDE. Before we proceed to study properties of q_m and r_m we need to consider an important example:

Example 1. Suppose that $a = 0$ and $\sigma = I$, then $X_t = W_t + x$. In this case

$$q_m(s, t_1, \ldots, t_n, t, x, y) = \mathbf{1}_{\{s<t_1<\cdots<t_n<t\}} H_m \times \left(\frac{y-x}{\sqrt{t-s}}\right) (2\pi)^{-d/2} (t-s)^{-(d+n)/2} e^{-\|y-x\|^2/2(t-s)}$$

where

$$H_m(z) = (-1)^n e^{\|z\|^2/2} \frac{\partial}{\partial z_{m_1}} \cdots \frac{\partial}{\partial z_{m_n}} e^{-\|z\|^2/2}$$

are multidimensional Hermite polynomials. This formula can be easily found using (13) and integration by parts. Also it appeared in several papers dealing with Itô–Wiener expansions, usually in a slightly different form (for example it can be found in some recent papers, where it was used to define local time [1,12]).

Note that in this example q_m is unbounded only in the neighbourhood of $t = s, x = z$ and does not depend on $t_1, \ldots, t_n$ (as long as $s < t_1 < \cdots < t_n < t$ holds) which is better than in the estimate (14).

In the following we will improve (14) for the case when a and b belong uniformly to $C_b^\infty(\mathbb{R}^d)$ and show that in this general situation the behaviour of kernels (or more precisely their singularity with respect to the variables s,t_k,t) is the same as in the example above.

4. Estimates for fundamental solution

In this section we use a modification of a variant of the standard parametrix method (the method that was used to prove upper bounds for the fundamental solution in [3]) to obtain some new estimates for the fundamental solution. The key difference between the classical method and our results is that our new estimates may be applied to q_m. This section contains some technical details of non-probabilistic nature, so it is worth noting that only the formulations of Theorems 5 and 6, Remark 1, Corollary 1 and related definitions of Q_u and $Q_{u,n}$ are needed for the rest of the article.

In the following we suppose that all elements of b and a belong uniformly to $C_b^\infty(\mathbb{R}^d)$ with constants $M_q, q = (q_1, \ldots, q_n), q_i = 1, \ldots, d$ (as defined by (6)). It is well-known that under this condition (together with strict ellipticity) we can have a Gaussian-type estimates for the derivatives of the fundamental solution G. These estimates are formulated in [10] without specifying constant dependence on coefficients. All the ideas and essentially all the proofs (but not the exact statement) for the following theorem and associated lemmas are contained in [3].

Theorem 5. *Suppose that a and b are continuous, all elements of $a(s,\cdot)$ and $b(s,\cdot)$ belong uniformly to $C_b^\infty(\mathbb{R}^d)$ with constants M_q and b satisfies (2). Then there exists constants $\gamma_k > 0$*

and $C_k > 0$, $k = 0, 1, \dots$ *depending only on* d, δ *and* M_q *such that for all* $0 < s < t < 1$, *non-negative integers* m, l, k *and* $p = (p_1, \dots, p_m) \in \{1, \dots, d\}^m$, $q = (q_1, \dots, q_l) \in \{1, \dots, d\}^l$, $r = (r_1, \dots, r_k) \in \{1, \dots, d\}^k$*:*

$$\left|\left(\frac{\partial}{\partial x_{r_1}} + \frac{\partial}{\partial y_{r_1}}\right) \dots \left(\frac{\partial}{\partial x_{r_k}} + \frac{\partial}{\partial y_{r_k}}\right) \frac{\partial}{\partial x_{p_1}} \dots \frac{\partial}{\partial x_{p_m}} \frac{\partial}{\partial y_{q_1}} \dots \frac{\partial}{\partial y_{q_l}} G(s, t, x, y)\right|$$
$$\leqslant C_{m+l+k}(t-s)^{-(d+m+l)/2} \exp^{-\gamma_{m+l+k} \frac{\|x-y\|^2}{2(t-s)}} . \tag{19}$$

Since, to our knowledge, this exact version of the Gaussian estimates for the fundamental solution of parabolic PDE is not present anywhere, we feel that it is appropriate to provide a brief version of the proof, even though this proof is mostly the repetition of a slightly modified version of the parametrix method from [3]. It is also appropriate since ideas used here are also useful for estimating kernels q_m as we will show later. We proceed with few lemmas to prepare the proof of Theorem 5.

Denote as $R_u = R_u(C, \gamma)$ a set of real-valued functions $F(s, t, x, y)$ on $\{0 \leqslant s \leqslant t \leqslant 1\} \cup \{x, y \in \mathbb{R}^d\}$ that satisfy:

$$|F(s, t, x, y)| \leqslant C(t-s)^{-(d+u)/2}(2\pi)^{-d/2} \exp^{-\gamma \frac{\|x-y\|^2}{2(t-s)}}$$

with some positive constants C, γ. Define a set $Q_u = Q_u(C, \gamma)$, where $C = (C_0, C_1, \dots)$ and $\gamma = (\gamma_0, \gamma_1, \dots)$ are sequences of positive constants, as a set of functions $F(s, t, x, y)$ defined for $0 \leqslant s \leqslant t \leqslant 1$ and $x, y \in \mathbb{R}^d$, such that $Q_u \subset C_b^\infty(\mathbb{R}^d)$ and:

$$\begin{aligned} Q_u(C, \gamma) = \Bigg\{ & F(s, t, x, y) | \\ & \forall k, l, m = 0, 1, \dots; \forall p = (p_1, \dots, p_m), \forall q = (q_1, \dots, q_l), \\ & \forall r = (r_1, \dots, r_k); p_i, q_i, r_i \in \{1, \dots, d\} : \\ & \left(\frac{\partial}{\partial x_{r_1}} + \frac{\partial}{\partial y_{r_1}}\right) \dots \left(\frac{\partial}{\partial x_{r_k}} + \frac{\partial}{\partial y_{r_k}}\right) \frac{\partial}{\partial x_{p_1}} \dots \frac{\partial}{\partial x_{p_m}} \frac{\partial}{\partial y_{q_1}} \dots \frac{\partial}{\partial y_{q_l}} F \in \\ & R_{u+m+l}(C_{m+l+k}, \gamma_{m+l+k}) \Bigg\}. \end{aligned}$$

This definition allows us to write a sequence of estimates in a simple form. For example the conclusion of Theorem 5 now can be rewritten as follows: $G \in Q_0(C, \gamma)$, where C and γ depend only on d,δ and M_q.

The following Lemma 1 is the main tool for obtaining Gaussian estimates. A different version of this lemma (for more general case but with weaker conclusion) with similar proof is present in [3] (Lemma 9.1 on p. 94). We only sharpened the statement of the lemma to fit our purposes.

Lemma 1. *Suppose that* $F_1 \in Q_u(C^1, \gamma^1)$, $F_2 \in Q_v(C^2, \gamma^2)$ *for* $u < 2$, $v < 2$. *and*

$$F_3(s, t, x, y) = \int_s^t \int_{\mathbb{R}^d} F_1(s, r, x, z) F_2(r, t, z, y) dz dr$$

then $F_3 \in Q_{u+v-2}(C^3, \gamma^3)$, *where* C^3 *and* γ^3 *depend only on* C^1, γ^1, C^2, γ^2, u, v.

Proof. Directly estimating F_1 and F_2 under the integral we obtain $F_3 \in R_{u+v-2}$. To deal with the derivatives of F_3 we need a representation of the derivatives of

$$H(s,t,x,y,r) = \int_{\mathbb{R}^d} F_1(s,r,x,z)F_2(r,t,z,y)dz.$$

Since we know that $F_1(s,r,x,z)F_2(r,t,z,y)$ (and all their derivatives with respect to x,y,z) decrease exponentially to 0 as $|z| \to +\infty$ we can integrate by parts to show that:

$$\begin{aligned}\frac{\partial}{\partial x_i} H(s,t,x,y,r) &= \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial z_i}\right) F_1(s,r,x,z)F_2(r,t,z,y)dz \\ &\quad + \int_{\mathbb{R}^d} F_1(s,r,x,z)\frac{\partial}{\partial z_i} F_2(r,t,z,y)dz. \end{aligned} \tag{20}$$

Similar formula can be obtained for the derivative with respect to y. Adding these two representations we get:

$$\begin{aligned}\left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i}\right) H(s,t,x,y,r) &= \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial z_i}\right) F_1(s,r,x,z)F_2(r,t,z,y)dz \\ &\quad + \int_{\mathbb{R}^d} F_1(s,r,x,z)\left(\frac{\partial}{\partial z_i} + \frac{\partial}{\partial y_i}\right) \\ &\quad \times F_2(r,t,z,y)dz. \end{aligned} \tag{21}$$

We can see that (again by estimating directly)

$$\left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i}\right) H(s,t,x,y,r) \in R_{u+v-2}.$$

Using induction we can deal with all derivatives of this form.

Unfortunately direct estimate does not work for the derivative with respect to x since $\frac{\partial}{\partial x_i} F_1$ belongs to R_{v+1} and generally speaking it is not integrable with respect to dr if $v+1 > 2$. However we can use the following trick (which was also used in [3]): if $r \in [\frac{t+s}{2}, t]$ we estimate $\frac{\partial}{\partial x_i} F_1(s,r,x,z)$ directly and if $r \in [s, \frac{t+s}{2}]$ we estimate representation (20). We obtain that $\frac{\partial}{\partial x_i} F_3 \in R_{u+v-1}$. The same approach works for the derivative with respect to y. To prove the estimates for higher derivatives using the same trick we just need to prove an analogue of (20) for higher derivatives, which is easy to do after applying (20) several times consequently. It is obvious that constants in all estimates depend only on C^1,γ^1,C^2,γ^2,u,v. $\square$

The following lemma provides that solution of integral equation found by iteration have necessary estimates. The ideas leading to the proof of this lemma are present in [3] (pp. 99–101). Again we are only adapting corresponding results therein to our case.

Lemma 2. *Suppose that $F_1 \in Q_u(C^1,\gamma^1)$, $F_2 \in Q_v(C^2,\gamma^2)$, where $u < 2$, $v < 2$, Consider the following integral equation with unknown function $H(s,t,x,y)$, which holds for all $0 \leqslant s < t \leqslant 1$; $x, y \in \mathbb{R}^d$:*

$$H(s,t,x,y) = F_1(s,t,x,y) + \int_s^t \int_{\mathbb{R}^d} F_2(s,r,x,z)H(r,t,z,y)dzdr.$$

Then there is a solution to this equation that belongs to $Q_u(C^3,\gamma^3)$, where C^3 and γ^3 depend only on C^1, γ^1, C^2, γ^2, u, v.

Proof. Denote $K_0 = F_1$, and

$$K_{i+1}(s,t,x,y) = \int_s^t \int_{\mathbb{R}^d} F_2(s,r,x,z) K_i(r,t,z,y)dzdr$$

for $i = 0, 1, \ldots$. The solution to the integral equation can be written as:

$$H = \sum_{i=0}^{\infty} K_i \tag{22}$$

where the sum converges pointwise to some function in R_u (this is a classical result from the parametrix method, which follows from direct estimates of each K_i).

We notice that solution of our integral equation can also be found as

$$H(s,t,x,y) = F_1(s,t,x,y) + \int_s^t \int_{\mathbb{R}^d} H_0(s,r,x,z) F_1(r,t,z,y)dzdr$$

where H_0 satisfies the same integral equation with F_1 replaced by F_2. But we already know that we can find a solution H_0 that belongs to R_v, since $F_2 \in R_v$, $v < 2$. If we prove that in fact $H_0 \in Q_v$ it would be enough to finish the proof, since from Lemma 1 we immediately obtain that $H \in Q_u$.

To show this we notice that H_0 satisfies the following equation (the idea is taken from [3] p.101)

$$\begin{aligned} H_0(s,t,x,y) = {} & \sum_{i=0}^{2l+1} K_i^0(s,t,x,y) + \int_s^t \int_{\mathbb{R}^d} K_l^0(s,r_1,x,z_1) \\ & \times \int_{r_1}^t \int_{\mathbb{R}^d} H_0(r_1,r_2,z_1,z_2) K_l^0(r_2,t,z_2,y) dz_2 dr_2 dz_1 dr_1 \end{aligned}$$

where $K_0^0 = F_2$ and

$$K_{i+1}^0(s,t,x,y) = \int_s^t \int_{\mathbb{R}^d} F_2(s,r,x,z) K_i^0(r,t,z,y)dzdr.$$

Using Lemma 1 we can see that $K_i^0 \in Q_{v+i(v-2)}$, so we only need to estimate the last integral. We can choose l to depend on n, v, such that $v + l(v-2) + n < 0$. Then we have $K_l^0 \in R_{-n}$ and consequently the integral can be differentiated n times with respect to x, y, the derivative can be taken under the integral and the whole integral is a function at least in R_v (since we already know that $H_0 \in R_v$), which is enough to complete the proof. □

Proof of Theorem 5. It is well-known that G can be represented as

$$G(s,t,x,y) = G_0(s,t,x-y,y) + \int_s^t \int_{\mathbb{R}^d} G_0(s,r,x-z,z) H(r,t,z,y)dzdr$$

where $G_0(s,t,x,y)$ is a fundamental solution of (5) with the coefficients of the operator $L^{s,x}$ being "frozen" at point $x = y$, i.e. we replace $L^{s,x}$ by

$$L^{s,x,y} f(x) = \sum_{i=1}^d a_i(s,y) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^d (\sigma_{ik}\sigma_{jk})(s,y) \frac{\partial^2 f}{\partial x_i \partial x_j}(x).$$

The function H is a solution of the integral equation

$$H(s,t,x,y) = F(s,t,x,y) + \int_s^t \int_{\mathbb{R}^d} F(s,r,x,z)H(r,t,z,y)dzdr$$

where

$$F(s,t,x,y) = (L^{s,x,y} - L^{s,x})G_0(s,t,x-y,y)$$

According to formula (6) on p. 66 from [3] $G_0(s,t,x-y,y) \in Q_0$ with constants depending only on d,δ and M_q (for our case G_0 is a Gaussian density). Using this, it is easy to verify that $F \in Q_1$ with constants depending only on d,δ and M_q. From Lemmas 1 and 2 we obtain that $H \in Q_0$ and consequently $G \in Q_0$ with constants depending only on d,δ and M_q. $\square$

Now we are going to extend Lemma 1 so that we can estimate q_m.

Denote for non-negative integer n:

$$Q_{u,n}(C,\gamma) = \left\{ F = \sum_{|p|+|q|\leqslant n} D_q^x D_p^y F_{p,q} : F_{p,q} \in Q_{u-n}(C,\gamma) \right\}$$

where D_q^x are derivatives with respect to x_{q_i} with multiindex $q = (q_1, \ldots, q_m)$ and $m = |q|$ is the number of the derivatives taken. It is obvious that $Q_{u,n} \subset Q_u$, but the converse statement is not true (for $n > 0$). To prove it we may take Gaussian density multiplied by $(t-s)^{-u/2}$ as an example

$$F(s,t,x,y) = (t-s)^{-(d+u)/2}(2\pi)^{-d/2} e^{-\frac{\|x-y\|^2}{2(t-s)}}$$

This is obviously an element of Q_u. Suppose that the representation from the definition of $Q_{u,n}$ is possible and

$$F(s,t,x,y) = \sum_{|p|+|q|\leqslant n} D_q^x D_p^y F_{p,q}(s,t,x,y),\ F_{p,q} \in Q_{u-n}(C,\gamma).$$

Integrating with respect to dx we obtain

$$(t-s)^{-u/2} = \int_{\mathbb{R}^d} \sum_{|p|\leqslant n} \left(\frac{\partial}{\partial x_{p_1}} + \frac{\partial}{\partial y_{p_1}} \right) \ldots \left(\frac{\partial}{\partial x_{p_k}} + \frac{\partial}{\partial y_{p_k}} \right) F_{p,0}(s,t,x,y)dx$$

where $p = (p_1, \ldots, p_k)$. Note that the derivatives with respect to x vanish under the integral and therefore derivatives with respect to y can be represented using the sum of derivatives with respect to x and y. The function on the right hand side is an integral of the element of R_{u-n}, which means it does not exceed $(t-s)^{-(u-n)/2} = (t-s)^{-u/2}(t-s)^{n/2}$. This is a contradiction and it means that the function F in our example does not belong to $Q_{u,n}$.

It turns out that after we replace Q_u by $Q_{u,n}$ in Lemma 1 we need only that $u - n < 2$ rather than $u < 2$ (and the same for v, m).

Lemma 3. *Suppose that $F_1 \in Q_{u,n}(C^1,\gamma^1)$, $F_2 \in Q_{v,m}(C^2,\gamma^2)$ for $n = 0,1,\ldots$; $m = 0,1,\ldots$; $u < n+2$; $v < m+2$ and*

$$F_3(s,t,x,y) = \int_s^t \int_{\mathbb{R}^d} F_1(s,r,x,z)F_2(r,t,z,y)dzdr.$$

Then $F_3 \in Q_{u+v-2,n+m}(C^3,\gamma^3)$, where C^3 and γ^3 depend only on C^1, γ^1, C^2, γ^2, u, v, n, m.

Proof. We know that if $n = 1$:

$$F_1 = F_{1,0} + \sum_{i=1}^{d} \left(\frac{\partial}{\partial x_i} F^x_{1,i} + \frac{\partial}{\partial y_i} F^y_{1,i} \right)$$

where $F_{1,0}, F^x_{1,i}, F^y_{1,i} \in Q_{u-1}$ and $u - 1 < 2$. Note that (this is a variant of (20))

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{\partial}{\partial z_i} F^y_{1,i}(s, r, x, z) F_2(r, t, z, y) dz = &-\frac{\partial}{\partial x_i} \int_{\mathbb{R}^d} F^y_{1,i}(s, r, x, z) F_2(r, t, z, y) dz \\ &+ \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial z_i} \right) F^y_{1,i}(s, r, x, z) \\ &\times F_2(r, t, z, y) dz.\end{aligned}$$

This immediately means that we found a desired representation of F_3 in the case $n = 1, m = 0$ (Lemma 1 is applied here). For the arbitrary n, m we can find the appropriate representation using iterations of the formula above and its analogue with interchanged variables y and x. □

If we reduce the highest limit for u, v by 2, we can drop the integral with respect to dr.

Lemma 4. *Suppose that $F_1 \in Q_{u,n}(C^1, \gamma^1)$, $F_2 \in Q_{v,m}(C^2, \gamma^2)$ for $n = 0, 1, \ldots$; $m = 0, 1, \ldots$; $u \leqslant n$; $v \leqslant m$ and*

$$F_3(s, t, x, y; r) = \int_{\mathbb{R}^d} F_1(s, r, x, z) F_2(r, t, z, y) dz, \quad r \in [s, t].$$

Then $F_3 \in Q_{u+v,n+m}(C^3, \gamma^3)$ ($F_3 = 0$ for $r \notin [s, t]$), where C^3 and γ^3 depend only on C^1, γ^1, C^2, γ^2, u, v, n, m.

Proof. If we prove that $F_1 \in Q_u$, $F_2 \in Q_v$ for $u, v \leqslant 0$ provides $F_3 \in Q_{u+v}$, then we may proceed like in the proof of Lemma 4. This statement can be verified by repeating the proof of Lemma 1, since the same approach works here as well. □

Now we are ready to estimate q_m.

Theorem 6. *Let $R^s_1, \ldots, R^s_n$ be a set of differential operators of orders $r_1, \ldots, r_n$ respectively with measurable coefficients that depend on parameter $s \in [0, 1]$. Suppose that a and b are continuous, all elements of $a(s, \cdot)$ and $b(s, \cdot)$ and coefficients of R^s_i belong uniformly to $C^\infty_b(\mathbb{R}^d)$ with constants M_q, and b satisfies (2). Then for all $0 < s < t_1 < \cdots < t_n < t < 1$ operator $T_{s,t_1} R^{t_1}_1 T_{t_1,t_2} \ldots R^{t_n}_n T_{t_n,t}$, acting on $C^\infty_b(\mathbb{R}^d)$, can be represented using a kernel $K(s, t_1, \ldots, t_n, t, x, y)$ with x, y as kernel variables. This kernel belongs to $Q_{r,r}(C, \gamma)$ with respect to the variables s, t, x, y. Constants C, γ depend only on M_q, δ, r, d, where $r = r_1 + \cdots + r_n$.*

Proof. We recall that by Theorem 5 we have $G \in Q_0$ and so its derivative of order k belongs to $Q_{k,k}$. The kernel K can be written using a repeated convolution of the kernel G of $T_{s,t}$ and its derivatives (the integrals and derivatives exist and can be taken in any order since $G \in Q_0$). To estimate K using Lemma 4 we only need to check that $G(s, t, x, y) f(t, y) \in Q_0$, if $f(t, \cdot)$ belongs uniformly to $C^\infty_b(\mathbb{R}^d)$. But this can be easily verified, for example:

$$\left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i}\right)(G(s,t,x,y)f(t,y)) = \left(\left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial y_i}\right)G(s,t,x,y)\right)f(t,y) + G(s,t,x,y)\frac{\partial}{\partial y_i}f(t,y) \in R_0. \quad \square$$

Remark 1. Suppose that we have a set of operators L_i, $i = 0, \ldots, n$ each satisfying the same conditions as L and corresponding $T_{i,s,t}$ for L_i defined as $T_{s,t}$ for L. Then if we replace $T_{t_i,t_{i+1}}$ by $T_{i,t_i,t_{i+1}}$ (T_{s,t_1} is replaced by T_{0,s,t_1} and $T_{t_n,t}$ – with $T_{n,t_n,t}$) the statement of Theorem 6 would also hold. It follows from the proof, since we never used the fact that we have the same coefficients, but only that the fundamental solutions have the same estimates.

Corollary 1. *Suppose that a and b are continuous, the elements of $a(s,\cdot)$ and $\sigma(s,\cdot)$ belong uniformly to $C_b^\infty(\mathbb{R}^d)$ with constants M_q, and b satisfies (2). Then $q_m(s, t_1, \ldots, t_n, t, x, y)$, where $s < t_1 < \cdots < t_n < t$, belongs to $Q_{n,n}(C,\gamma)$ with respect to variables s, t, x, y. Constants C, γ depend only on M_q, δ, n, d.*

Proof. This is a direct application of Theorem 6. $\square$

5. Examples of Itô–Wiener expansion kernels for a non-smooth σ

In the previous section we saw that the kernels q_m have the desired behaviour as long as the elements of σ (among other conditions) belong to $C_b^\infty(\mathbb{R}^d)$. We want to show that dropping this requirement may lead to the violation of the inequality provided by the statement $q_m \in R_n$ from Corollary 1. This is the main reason to look at the following examples.

Suppose that Eq. (1) has zero drift and σ is an orthogonal matrix: $a = 0$, $b = \sigma\sigma^T = I$. As long as σ is measurable such stochastic equation has a weak solution that is also a Wiener process. In other words we have a probability space where both X_t and W_t are Wiener processes. On this probability space we have two essentially different (generally speaking) Gaussian spaces generated by X and W. In some sense we may say that W generates Gaussian space that is a nonlinear transformation of Gaussian space generated by X (σ-algebra generated by W is smaller than one generated by X, because we can reverse equation to find W from X). This transformation is defined by some nonlinear function on Gaussian space that conserves Gaussian measure. In the case when X_t is a strong solution this function can be inverted. We are interested in the behaviour of the Itô–Wiener expansion of a function of X_t defined by our scheme using Gaussian space generated by W_t.

Let us show a simple example where $q_m \in R_n$ does not hold.

Example 2. Let $d = 1$ and

$$\sigma(x) = \begin{cases} 1, x \geqslant 0 \\ -1, x < 0 \end{cases}$$

This is well-known example that demonstrates the absence of strong solution.

Since it satisfies the requirements of Theorem 1 we have representation (13) for q_m. Let us look at q_1:

$$q_1(0, t_1, t, x, y) = \int_{\mathbb{R}} p(t_1, x - z)\sigma(z)\frac{\partial}{\partial z}p(t - t_1, z - y)dz$$

where $p(t,x) = (2\pi t)^{-d/2} e^{-\|x\|^2/2t}$ is standard Gaussian density with zero mean and variance t. Now if we take $x = y = 0$ we get:

$$q_1(0, t_1, t, 0, 0) = \int_{\mathbb{R}} p(t_1, z)\sigma(z)\frac{\partial}{\partial z} p(t - t_1, z)dz$$
$$= -(2\pi t)^{-1/2}(t - t_1)^{-1} \int_{\mathbb{R}} |z| p\left(\frac{t_1(t - t_1)}{t}, z\right) dz = M t^{-1}\sqrt{\frac{t_1}{t - t_1}}$$

where M is some negative constant. It is obvious that q_1 does not belong to R_1 uniformly on $s < t_1 < t$.

For the local time kernel r_1 we have:

$$r_1(0, t_1, t, 0, 0) = M \int_{t_1}^{t} s^{-1}\sqrt{\frac{t_1}{s - t_1}} ds = 2M \operatorname{arctg} \sqrt{\frac{t - t_1}{t_1}}$$

which means it is bounded.

Now we can compare this case to the situation when $q_1 \in R_1$ holds, which means that we have

$$|q_1(0, t_1, t, 0, 0)| \leqslant C_1 t^{-1}$$

and consequently

$$|r_1(0, t_1, t, 0, 0)| \leqslant C_1 ln \frac{t}{t_1}.$$

If $t_1 \to 0+$ then the behaviour in our example is better (in the example we have that r_1 is bounded, while $ln\frac{t}{t_1}$ is not). But if $t_1 \to t-$ then the estimate provides faster convergence to 0 than in our example, since:

$$\ln \frac{t}{t_1} \sim \frac{t}{t_1} - 1; \quad \operatorname{arctg} \sqrt{\frac{t - t_1}{t_1}} \sim \sqrt{\frac{t}{t_1} - 1}.$$

It may seem that the absence of strong solution causes this difference in the behaviour of the kernels, however this is not the case. In fact the real reason is the non-smoothness of σ. To confirm that we need to consider more complicated example.

Example 3. Let $d = 2$ and

$$\sigma(x) = \begin{cases} I, & x_1 < -1 \\ \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}, & x_1 > 0 \\ \begin{pmatrix} \cos\left(\frac{\pi(x_1+1)}{4}\right) & \sin\left(\frac{\pi(x_1+1)}{4}\right) \\ -\sin\left(\frac{\pi(x_1+1)}{4}\right) & \cos\left(\frac{\pi(x_1+1)}{4}\right) \end{pmatrix}, & x_1 \in [-1, 0]. \end{cases}$$

We defined σ such that $\sigma_{1,1}$ is a function only of the first variable smooth everywhere except $x_1 = 0, -1$ and in these points the function itself and its first derivative are continuous, except the derivative has a positive jump at $x_1 = 0$. Note that in this case σ satisfies Lipschitz condition and our X_t is in fact a strong solution. Let us look at $q_{(1,1)}$ with $x = y = 0$:

$$
\begin{aligned}
q_{(1,1)}(0, t_1, t_2, t, 0, 0) = \int_{\mathbb{R}^4} & p(t_1, z_1) \left(\sigma_{1,1}(z_1) \frac{\partial}{\partial z_{1,1}} + \sigma_{2,1}(z_1) \frac{\partial}{\partial z_{1,2}} \right) \\
& \times p(t_2 - t_1, z_2 - z_1) \\
& \times \left(\sigma_{1,1}(z_2) \frac{\partial}{\partial z_{2,1}} + \sigma_{2,1}(z_2) \frac{\partial}{\partial z_{2,2}} \right) p(t - t_2, z_2) dz_1 dz_2.
\end{aligned}
$$

We used the following notation for the coordinates: $z_1 = (z_{1,1}, z_{1,2}), z_2 = (z_{2,1}, z_{2,2})$. The function under the integral is a product of the function of $z_{1,1}, z_{2,1}$ and the function of $z_{1,2}, z_{2,2}$. Therefore the integral is the product of two integrals. After opening the brackets the integral splits into the sum of four integrals. It is easy to check that integrals with exactly one derivative of the second coordinate are zero, i.e two members of the sum are zero. The integral with two derivatives with respect to the second coordinate is easy to estimate: the integral with respect to the first coordinates is bounded by constant multiplied by $t^{-1/2}$ and the integral with respect to the second coordinates is equal to non-zero constant multiplied by $t^{-3/2}$. Let us study the remaining integral:

$$
\begin{aligned}
q_{(1,1)}(0, t_1, t_2, t, 0, 0) = \int_{\mathbb{R}^2} & p(t_1, z_{1,1}) \sigma_{1,1}(z_1) \frac{\partial}{\partial z_{1,1}} p(t_2 - t_1, z_{2,1} - z_{1,1}) \sigma_{1,1}(z_2) \\
& \times \frac{\partial}{\partial z_{2,1}} p(t - t_2, z_{2,1}) dz_{1,1} dz_{2,1} \int_{\mathbb{R}^2} p(t_1, z_{1,2}) \\
& \times p(t_2 - t_1, z_{2,1} - z_{1,1}) p(t - t_2, z_{2,1}) dz_{1,2} dz_{2,2}.
\end{aligned}
$$

The second integral above is equal to non-zero constant multiplied by $t^{-1/2}$. We want to study the behaviour of the first integral as $t_1 \to t-$ (t_2 is also changing to stay inside interval (t_1, t)). Denote $x = z_{1,1} - z_{2,1}$, $y = z_{2,1}$, $\sqrt{t_2 - t_1} = \varepsilon_1$, $\sqrt{t - t_2} = \varepsilon_2$ and

$$
g(\varepsilon_1, \varepsilon_2, x, y) = \sigma_{1,1}(x + y) \sigma_{1,1}(y) p(t - \varepsilon_1^2 - \varepsilon_2^2, x + y).
$$

The first integral is equal to:

$$
\begin{aligned}
J = & \int_{\mathbb{R}^2} p(t_1, z_{1,1}) \sigma_{1,1}(z_1) \frac{\partial}{\partial z_{1,1}} p(t_2 - t_1, z_{2,1} - z_{1,1}) \sigma_{1,1}(z_2) \\
& \times \frac{\partial}{\partial z_{2,1}} p(t - t_2, z_{2,1}) dz_{1,1} dz_{2,1} \\
= & \varepsilon_1^{-2} \varepsilon_2^{-2} \int_{\mathbb{R}^2} g(\varepsilon_1, \varepsilon_2, x, y) p(\varepsilon_1^2, x) p(\varepsilon_2^2, y) xy dx dy \\
= & \varepsilon_1^{-2} \varepsilon_2^{-2} \int_{[0,+\infty]^2} (g(\varepsilon_1, \varepsilon_2, x, y) - g(\varepsilon_1, \varepsilon_2, -x, y) - g(\varepsilon_1, \varepsilon_2, x, -y) \\
& + g(\varepsilon_1, \varepsilon_2, -x, -y)) p(\varepsilon_1^2, x) p(\varepsilon_2^2, y) xy dx dy.
\end{aligned}
$$

We are interested in the behaviour of this integral as $(\varepsilon_1, \varepsilon_2) \to (0+, 0+)$. We know that the first derivative of $\sigma_{1,1}$ is bounded, and its second derivative is unbounded at zero (the first derivative has a positive jump). We can use it to prove the following statement: for any $M > 0$ we can find $\delta > 0$ such that for all $0 < x < \delta, 0 < y < \delta, 0 < \varepsilon_1^2 < t/3, 0 < \varepsilon_2^2 < t/3$:

$$
g(\varepsilon_1, \varepsilon_2, x, y) - g(\varepsilon_1, \varepsilon_2, -x, y) - g(\varepsilon_1, \varepsilon_2, x, -y) + g(\varepsilon_1, \varepsilon_2, -x, -y) \geqslant Mxy
$$

Indeed $\sigma_{1,1}(x)$ can be represented in the neighbourhood of $x = 0$ as a double integral of its second derivative plus a "jump" term of the form $a|x|$. Taking function $f(x, y) = |x + y|$ it is easy to check that:

$$\lim_{(x,y)\to(0+,0+)} \frac{f(x, y) - f(x, -y) - f(-x, y) + f(-x, -y)}{xy}$$
$$= \lim_{(x,y)\to(0+,0+)} \frac{4\min(x, y)}{xy} = +\infty$$

This is also true for $f(x, y) = \sigma_{1,1}(x + y)$ since multiplying by positive constant and adding a two times differentiable function changes nothing. Our function $g(\varepsilon_1, \varepsilon_2, x, y)$ has an additional multiplier, but its derivatives with respect to x, y are bounded uniformly on $0 < \varepsilon_1^2 < t/3, 0 < \varepsilon_2^2 < t/3$. Therefore the same statement holds for g as function from x, y uniformly on $0 < \varepsilon_1^2 < t/3, 0 < \varepsilon_2^2 < t/3$. Using it we obtain the following estimate:

$$\begin{aligned}
J &\geqslant M\varepsilon_1^{-2}\varepsilon_2^{-2} \int_{[0,\delta]^2} p(\varepsilon_1^2, x)p(\varepsilon_2^2, y)x^2y^2dxdy \\
&\quad - 2\sup\left|\frac{\partial g}{\partial x}\right| \varepsilon_1^{-2}\varepsilon_2^{-2} \int_{[0,\delta]\times[\delta,+\infty]} p(\varepsilon_1^2, x)p(\varepsilon_2^2, y)x^2ydxdy \\
&\quad - 2\sup\left|\frac{\partial g}{\partial y}\right| \varepsilon_1^{-2}\varepsilon_2^{-2} \int_{[0,\delta]\times[\delta,+\infty]} p(\varepsilon_1^2, x)p(\varepsilon_2^2, y)xy^2dxdy \\
&\quad - 4\sup|g|\varepsilon_1^{-2}\varepsilon_2^{-2} \int_{[\delta,+\infty]^2} p(\varepsilon_1^2, x)p(\varepsilon_2^2, y)xydxdy.
\end{aligned}$$

The expression on the right hand side converges to $M/4$ as $(\varepsilon_1, \varepsilon_2) \to (0+, 0+)$, which means J takes values greater than any constant in any neighbourhood of $(\varepsilon_1, \varepsilon_2) = (0, 0)$. As a result $q_{(1,1)}(0, t_1, t_2, t, 0, 0)$ is unbounded as $t_1 \to t-$. This is a contradiction to the statement that $q_{(1,1)}$ belongs to R_2 uniformly on t_1, t_2. This contradiction is obviously the consequence of the non-smoothness of σ.

6. Local time as a generalized functional

Note that the kernels of the Itô–Wiener expansion of $I^n_{LT}(\delta_x) = \lim_{\varepsilon\to0+} I^n_{LT}(f_\varepsilon)$, where f_ε converges weakly as measures to δ_x, can be represented by r_m. The estimates from Theorem 6 allow us to see that we can choose a good multiplier ψ in the definition (9) of the operator A to get rid of the singularities in r_m and make $\|(A^{\otimes n}r)_m\|_2$ finite for all $n \geqslant 1$ (for example we can take $\psi(t) = \frac{1}{2}t^\theta$ with large enough θ). Therefore the following result seems natural.

Theorem 7. *Suppose that a and b are continuous, the elements of $a(s, \cdot)$ and $\sigma(s, \cdot)$ belong uniformly to $C_b^\infty(\mathbb{R}^d)$ with constants M_q, b satisfies (2) and the operator A satisfies (9) and (10). If we additionally have that:*

$$\int_0^1 (M_{0,t}(\psi))^{n+1}t^{1-d}dt < +\infty \tag{23}$$

where

$$M_{s,t}(\psi) = \sup_{i,u\in[s,t]} \psi_i^2(u) \tag{24}$$

then there exists n-renormalized local time in $\Phi_2(A)$ of X_t at any point y.

The existence of n-renormalized local time in $\Phi_2(A)$ means the finiteness of $\Phi_2(A)$-norm for $I^n_{LT}(\delta_x)$, which is local time without the first $n+1$ members of its Itô–Wiener expansion (or more precisely the existence of the corresponding limit of the local time approximations, see Definition 2).

Unfortunately we cannot prove that $\Phi_2(A)$ norm is finite directly using Gaussian estimates from Theorem 6, since it includes controlling the dependence of the $\|(A^{\otimes n}r)_m\|_2$ on n and consequently finding the behaviour of all constants in our estimates as $n \to +\infty$. But the complicated nature of these estimates does not allow us to do that. Therefore we take a different approach and estimate the norm in $\Phi_2(A)$ as a whole instead of estimating each member of the corresponding series separately (which includes using (8) together with (11) instead of its alternative (7)). To achieve this we develop a representation of $\Phi_2(A)$ norm of $I_{LT}(f)$ and $I^n_{LT}(f)$, assuming that (9) and (10) hold. For this purpose we construct a new stochastic differential equation, such that the transition density of its solution determines $\Phi_2(A)$ norm of any functionals of X.

Consider a stochastic differential equation for a stochastic process Y_t in $\mathbb{R}^{2d}$:

$$dY_t = \tilde{a}(t, Y_t)dt + \tilde{\sigma}(t, Y_t)d\hat{W}_t \tag{25}$$

where $\hat{W}_t$ is a Wiener process in $\mathbb{R}^{3d}$ and

$$\tilde{a}(t,x) = (a(t,x^1), a(t,x^2))$$

$$\tilde{\sigma}(t,x) = \begin{pmatrix} \sigma(t,x^1) & 0 \\ 0 & \sigma(t,x^2) \end{pmatrix} \begin{pmatrix} \Lambda_1(\psi(t)) & 0 & \Lambda_2(\psi(t)) \\ 0 & \Lambda_1(\psi(t)) & \Lambda_2(\psi(t)) \end{pmatrix}$$

for any $t > 0$ and $x^1 \in \mathbb{R}^d$, $x^2 \in \mathbb{R}^d$, $x = (x^1, x^2)$ with $\Lambda_1(z)$ and $\Lambda_2(z)$ are diagonal matrices that depend on $z \in [-1,1]^d$ as follows

$$\Lambda_1(z)_{ii} = \sqrt{1-(z_i)^2}; \qquad \Lambda_2(z)_{ii} = z_i.$$

Let $Y_t = (Y^1_t, Y^2_t)$, where Y^1 and Y^2 are the projections on the subspaces spanned by the first d coordinates and the other d coordinates respectively. It is obvious that Y^1_t and Y^2_t are the solutions of the original Eq. (1) with two different Wiener processes. We also note that:

$$\tilde{b}(t,x) = \tilde{\sigma}(t,x)\tilde{\sigma}^T(t,x) = \begin{pmatrix} \sigma(t,x^1)\sigma^T(t,x^1) & \sigma(t,x^2)\Lambda_2((\psi(t))^2)\sigma^T(t,x^1) \\ \sigma(t,x^1)\Lambda_2((\psi(t))^2)\sigma^T(t,x^2) & \sigma(t,x^2)\sigma^T(t,x^2) \end{pmatrix}$$

is uniformly positive definite, or more precisely:

$$(\tilde{b}(t,x)\lambda, \lambda) > (1-\beta^2)\delta|\lambda|^2$$

where δ is constant from (2) and β is constant from (10). We can use Y_t to estimate $\Phi_2(A)$ norm for the functional $I^n_{LT}(f)$ due to the following lemma.

Lemma 5. *If $Y_0 = (X_0, X_0)$ and Eq. (1) has a unique strong solution X, then for any operator A satisfying (9) and (10) and any bounded functionals $F = F(X)$, $G = G(X)$ of the process X we have:*

$$(F(X), G(X))_{2,A} = EF(Y^1)G(Y^2).$$

Proof. This equality is a direct consequence of the formula (11) since the pair $X(W^1)$ and $X(W^2)$ have the same distribution as Y^1 and Y^2, where both W^1 and W^2 has the same definition as $\hat{W}$ from (11): with W being the same in both cases and $\tilde{W}$ being independent with each other and W. Indeed the Eq. (25) is actually two equations of the same form as (1) with two different Wiener processes, that have the same distribution, as the pair (W^1, W^2). □

Denote

$$\tilde{L}^{t,x} f(x) = \sum_{i=1}^{2d} \tilde{a}_i(t,x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i=1}^{2d} \sum_{j=1}^{2d} \sum_{k=1}^{3d} \tilde{\sigma}_{ik}(t,x) \tilde{\sigma}_{jk}(t,x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

and let $\tilde{G}(s,t,x,y)$ for $s < t$ be a fundamental solution of the following partial differential equation $\frac{\partial}{\partial t} u(t,x) + \tilde{L} u(t,x) = 0$ solved backwards in time, which is also the probability density with respect to variable y of the strong solution Y_t of the stochastic differential Eq. (25) with $Y_s = x$.

We can write $\Phi_2(A)$ norm for $I^n_{LT}(f)$ using $\tilde{G}$ and q_m from Krylov–Veretennikov representation.

Lemma 6. *Suppose that a and b are continuous, the elements of $a(s,\cdot)$ and $\sigma(s,\cdot)$ belong uniformly to $C_b^\infty(\mathbb{R}^d)$, b is uniformly continuous on $[0,1] \times \mathbb{R}^d$ and $Y_0 = (x,x), x \in \mathbb{R}^d$. Then for any operator A satisfying (9) and (10) and bounded continuous functions f, g we have:*

$$(I^n_{LT}(f), I^n_{LT}(g))_{2,A} = \int_0^1 \int_0^1 \int_{\mathbb{R}^{2d}} K_n(s,t,x,y^1,y^2) f(y^1) f(y^2) ds dt dy^1 dy^2 \tag{26}$$

where, if $s < t$ (case $t < s$ is symmetrical),

$$\begin{aligned} K_n(s,t,x,y^1,y^2) &= \int_{\mathbb{R}^d} \tilde{G}(0,s,(x,x),(y^1,u)) G(s,t,u,y^2) du \\ &\quad - \sum_{k=0}^{n} \sum_{|m|=k} \int_{\Delta_n[0,s]} q_m(0,t_1,\ldots,t_n,s,x,y^1) q_m \\ &\quad \times (0,t_1,\ldots,t_n,t,x,y^2) \prod_{i=1}^{n} \psi^2_{m_i}(t_i) dt_1 \ldots dt_n. \end{aligned} \tag{27}$$

Proof. Using Theorem 1 we can write:

$$I^n_{LT}(f) = I_{LT}(f) - \sum_{k=0}^{n} \sum_{|m|=k} \int_{\mathbb{R}^d} \int_0^1 \xi_m(s,y,W) ds f(y) dy$$

where

$$\xi_m(s,y,W) = \int_{\Delta_n[0,s]} q_m(0,t_1,\ldots,t_n,s,x,y) dW(t_1) \ldots dW(t_n)$$

where ξ_m is well-defined and the integral with respect to $f(y)dy$ can be taken after the stochastic integral due to the estimate from Corollary 1.

Lemma 5 allows us to find:

$$(I^n_{LT}(f), I^n_{LT}(g))_{2,A} = \int_0^1 \int_0^1 Ef(Y^1_s) g(Y^2_t) ds dt$$

$$- \sum_{k=0}^{n} \sum_{|m|=k} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 \int_0^1 E\xi_m(s, y^1, W^1) \times \xi_m(t, y^2, W^2) ds dt f(y^1) g(y^2) dy^1 dy^2.$$

Now, since:

$$E\xi_m(s, y^1, W^1)\xi_m(t, y^2, W^2) = \int_{\Delta_n[0,min(s,t)]} q_m(0, t_1, \ldots, t_n, s, x, y^1) q_m \times (0, t_1, \ldots, t_n, t, x, y^2) \prod_{i=1}^{n} \psi_{m_i}^2(t_i) dt_1 \ldots dt_n$$

we only have to find $Ef(Y_s^1)g(Y_t^2)$ using that the fundamental solution $\tilde{G}(s, t, x, \cdot)$ is a probability density of $(Y^1(t), Y^2(t))$ under the condition $(Y^1(s), Y^2(s)) = x$:

$$Ef(Y_s^1)g(Y_t^2) = \int_{\mathbb{R}^{4d}} \tilde{G}(0, s, (x, x), (y^1, u)) \tilde{G}(s, t, (y^1, u), (v, y^2)) f(y^1) g(y^2) dv du dy^1 dy^2.$$

But since Y^1 and Y^2 are the solutions of two unrelated equations with two different Wiener processes such that increments of one Wiener process are independent together with the increments of the other on non-intersecting intervals, we also have that

$$\int_{\mathbb{R}^d} \tilde{G}(s, t, (y^1, u), (v, y^2)) dv = G(s, t, u, y^2).$$

And this completes the proof of the lemma. □

Gaussian estimates for $\tilde{G}$ and q_m provide that kernel K_n is bounded by Gaussian density. But if we integrate this estimate with respect to $dsdt$ we get an unbounded function for $d > 1$. Therefore it does not allow us to define renormalized local time in $\Phi_2(A)$. However renormalized local time in $\Phi_2(A)$ was already defined in [1] and we can improve our estimates to define it for our case.

We want to control "renormalized" $\tilde{G}$, when s or t is close to 0 by including an additional dependence of constants in the estimate on ψ. To achieve this we need to find a specific representation of $\tilde{G}$ to deal with both renormalization and the dependence on ψ. Fortunately this can be done by deriving a Taylor's expansion of $\tilde{G}$ with respect to a multiplicator of ψ^2.

Denote by $\tilde{G}_q$ the fundamental solution $\tilde{G}$ if ψ is replaced by $\sqrt{q}\psi$ for $q \in [0, 1]$.

Theorem 8. *Suppose that a and b are continuous, the elements of $a(s, \cdot)$ and $\sigma(s, \cdot)$ belong uniformly to $C_b^\infty(\mathbb{R}^d)$ with constants M_p, b satisfies (2) and the operator A satisfies (9) and (10). Then for all $0 < s < t < 1$, $x = (x^1, x^2)$, $y = (y^1, y^2)$; $x^1, x^2, y^1, y^2 \in \mathbb{R}^d$ and $N = 0, 1, \ldots$:*

$$\tilde{G}(s, t, x, y) = \sum_{n=0}^{N} \sum_{|m|=n} \int_{\Delta_n[0,1]} q_m(s, t_1, \ldots, t_n, t, x^1, y^1) q_m \times (s, t_1, \ldots, t_n, t, x^2, y^2) \prod_{i=1}^{n} \psi_{m_i}^2(t_i) dt_1 \ldots dt_n + \int_0^1 \tilde{G}_q^{N+1}(s, t, x, y)(N + 1)(1 - q)^N dq \tag{28}$$

where q_m are the kernels of Krylov–Veretennikov representation given by (13),

$$\tilde{G}_q^{N+1}(s,t,x,y) = \int_{\Delta_{N+1}[0,1]} \int_{\mathbb{R}^{2d(N+1)}} \tilde{G}_q(s,t_1,x,z_1) R^{t_1,z_1} \tilde{G}_q(t_1,t_2,z_1,z_2) \dots \times \tilde{G}_q(t_N,t_{N+1},z_N,z_{N+1}) R^{t_{N+1},z_{N+1}} \times \tilde{G}_q(t_{N+1},t,z_{N+1},y) dz_1 \dots dz_{N+1} dt_1 \dots dt_{N+1} \tag{29}$$

for $N = 0, 1, \dots$ and

$$R^{t,x,y} = \sum_{i=1}^{d} \psi_i^2(t) S_i^{t,x} S_i^{t,y}.$$

Additionally $\tilde{G}_q^{N+1} \in Q_{0,2(N+1)}(\tilde{C},\gamma)$ with $\tilde{C} = (M_{s,t}(\psi))^{N+1} C$ where $M_{s,t}(\psi)$ is given by (24) and constants C, γ depend only on N, β, δ, M_p, d. In particular

$$|\tilde{G}_q^{N+1}(s,t,x,y)| \leqslant C_0 (M_{s,t}(\psi))^{N+1} (t-s)^{-d} (2\pi)^{-d} e^{-\gamma_0 \frac{\|x-y\|^2}{2(t-s)}}. \tag{30}$$

Proof. Notice that our conditions provide that we can apply Theorems 5 and 6 to the operator $\tilde{L}_q$, which is the operator $\tilde{L}$ for ψ replaced by $\sqrt{q}\psi$. So $\tilde{G}_q$, q_m and $\tilde{G}_q^{N+1}(s,t,x,y)$ are well-defined and possess Gaussian estimates with constants that do not depend on q and ψ if β is fixed (due to Theorem 6: $\tilde{G}_q^{N+1} \in Q_{0,2(N+1)}$). Moreover if we use Theorem 6 to estimate $\tilde{G}_q^{N+1}(s,t,x,y)$, such that at first we estimate the integral with respect to $dz_1 \dots dz_{N+1}$ without including the multiplication factors ψ_i^2 that come from operator $R^{t,z}$ and then estimate integrals with respect to $dt_1 \dots dt_{N+1}$, we obtain additional dependence on $M_{s,t}(\psi)$, in particular we can prove inequality (30).

Hence to prove the theorem we only need to show that for $q \in (0,1)$ there exists derivative

$$\frac{d^k}{dq^k} \tilde{G}_q(s,t,x,y) = k! \tilde{G}_q^k(s,t,x,y)$$

which is continuous for $q \in [0,1]$. Then formula (28) is just the Taylor expansion of $\tilde{G}_q$ with the residue in the integral form, since

$$\tilde{G}_0(s,t,x,y) = G(s,t,x^1,y^1) G(s,t,x^2,y^2)$$

and consequently for $k = 1, 2, \dots$ (we can pass to the limit $q \to 0+$ under the integral in (29) due to the estimate from Theorem 6)

$$\tilde{G}_0^k(s,t,x,y) = \int_{\Delta_n[0,1]} q_m(s,t_1,\dots,t_n,t,x^1,y^1) \times q_m(s,t_1,\dots,t_n,t,x^2,y^2) \prod_{i=1}^{n} \psi_{m_i}^2(t_i) dt_1 \dots dt_n$$

We notice that

$$\tilde{L}_q^{t,x} = \tilde{L}_0^{t,x} + q R^{t,x}$$

which allows us to derive a well-known identity (with $\varepsilon R^{t,x}$ as "perturbation"):

$$\tilde{G}_{q+\varepsilon}(s,t,x,y) = \tilde{G}_q(s,t,x,y) + \varepsilon \int_s^t \int_{\mathbb{R}^{2d}} \tilde{G}_{q+\varepsilon}(s,r,x,z) R^{r,z} \tilde{G}_q(r,t,z,y) dz dr.$$

Due to Remark 1 after Theorem 6 the integral in the right hand side taken only with respect to dz (and all its derivatives with respect to x, y) has Gaussian bound that does not depend on r,q and ε. Therefore the whole integral is bounded for $q, q+\varepsilon \in [0,1]$ and we can pass to the limit in q and $q+\varepsilon$ under the integral. Using these properties and the formula above we obtain that $\tilde{G}_q$ is continuous for $q \in [0,1]$. Furthermore we have that the integral is continuous for $q, q+\varepsilon \in [0,1]$ and consequently $\tilde{G}_q$ is differentiable with the derivative

$$\tilde{G}_q^1(s,t,x,y) = \int_s^t \int_{\mathbb{R}^{2d}} \tilde{G}_q(s,t_1,x,z_1) R^{t_1,z_1} \tilde{G}_q(t_1,t,z_1,y) dz_1 dt_1$$

which is also continuous for $q \in [0,1]$. Repeating this procedure for higher derivatives with induction and iteration of the "perturbation" formula above is enough to prove the theorem. □

Remark 2. Note that this theorem actually allow us to prove Krylov–Veretennikov representation for our case. Let ξ and η be two random variables measurable with respect to our Wiener process W (defined for $t \in [0,1]$) and let their Itô–Wiener expansions be given by two sets of kernels a_m^ξ and a_m^η respectively. Then, by the definition of $\Gamma(A)$, we have:

$$\begin{aligned} E(\Gamma(A)\xi\Gamma(A)\eta) &= \sum_{n=0}^{+\infty} \sum_{|m|=n} \int_{\Delta_n[0,1]} a_m^\xi(t_1,\ldots,t_n) \\ &\quad \times a_m^\eta(t_1,\ldots,t_n)\psi_{m_1}^2(t_1)\ldots\psi_{m_n}^2(t_n)dt_1\ldots dt_n \end{aligned}$$

and on the other hand, if $\xi = f(X_t)$ and $\eta = g(X_t)$:

$$\begin{aligned} E(\Gamma(A)\xi\Gamma(A)\eta) &= E(f(Y_t^1)g(Y_t^2)) \\ &= \int_{\mathbb{R}^{2d}} \tilde{G}(0,t,(x,x),(y^1,y^2)) f(y^1) g(y^2) dy^1 dy^2 \end{aligned}$$

where X_t and Y_t are the solutions of the respective equations with $X_0 = Y_0^1 = Y_0^2 = x$. We know that $\tilde{G}$ can be represented using (28) and therefore we can use the same approach as in the theorem above (having ψ replaced by some $\sqrt{q}\psi$) to show that functions q_m are connected to the Itô–Wiener expansion kernels a_m as in Theorem 1. This allows us to drop the condition of uniform continuity for b in Lemma 6 and consequently in Theorem 7. Of course our conditions on the coefficients here are much more strict than in the original result by Krylov and Veretennikov.

Now we are ready to prove our main result concerning the existence of local time in the space $\Phi_2(A)$.

Proof of Theorem 7. From Lemma 6 we know that the kernel K_n that describes the norm in $\Phi_2(A)$ is given by (27). Using Theorem 8 we replace $\tilde{G}$ by its representation (28). We obtain two sums with q_m which are the same with different signs since the integral of $q_m(0, t_1,\ldots,t_n,s,x,u)$ multiplied by $G(s,t,u,y^2)$ with respect to du is $q_m(0,t_1,\ldots,t_n,t,x,y^2)$ (it follows from (13), the definition of q_m). We have for $s < t$:

$$\begin{aligned} &K_n(s,t,x,y^1,y^2) \\ &\quad = \int_{\mathbb{R}^d} \int_0^1 \tilde{G}_q^{n+1}(0,s,(x,x),(y^1,u)) G(s,t,u,y^2)(n+1)(1-q)^n dq du. \end{aligned}$$

Using Gaussian estimates for $\tilde{G}_q^{n+1}(0, s, (x, x), (y^1, u))$ and $G(s, t, u, y^2)$ (from Theorems 5 and 8) we find that for $s < t$:

$$|K_n(s, t, x, y^1, y^2)| \leqslant C(M_{0,s}(\psi))^{n+1}(ts)^{-d/2}(2\pi)^{-d}e^{-\gamma(\frac{\|y^1-x\|^2}{2s}+\frac{\|y^2-x\|^2}{2t})}$$

The condition (23) provides that

$$\int_0^1\int_0^1 |K_n(s, t, x, y^1, y^2)|dsdt \leqslant C\int_0^1\int_0^t (M_{0,s}(\psi))^{n+1}(ts)^{-d/2}dsdt$$
$$= C\int_0^1 (M_{0,s}(\psi))^{n+1}s^{1-d}ds < +\infty$$

and consequently

$$\int_0^1\int_0^1 K_n(s, t, x, y^1, y^2)dsdt$$

is bounded and continuous function of x,y^1,y^2.

It means that if f and g converge weakly as measures to δ-measure at some point x then the limit of the $(I^n_{LT}(f), I^n_{LT}(g))_{2,A}$ exists and finite, i.e. the limit of $I^n_{LT}(f)$ exists and local time exists in $\Phi_2(A)$. $\square$

Note that in fact we can prove that the kernel

$$\int_0^1\int_0^1 K_n(s, t, x, y^1, y^2)dsdt$$

is continuously differentiable l times (with arbitrary but fixed l) if we suppose that the appropriate condition on $M_{0,t}(\psi)$ holds (more strict version of (23), that depends on l). Therefore we can show the existence of another version of local time, where δ-measure is replaced by a generalized function of order l from the space $S^*(\mathbb{R}^d)$. Similar result for Wiener process was proven in [1].

References

[1] A.A. Dorogovtsev, V.V. Bakun, Random mappings and a generalized additive functional of a wiener process, Theory of Probability and its Applications 48 (2003).
[2] M. Eddahbi, R. Lacayo, J.L. Sole, J. Vives, C.A. Tudor, Renormalization of the local time for the d-dimensional fractional brownian motion with n parameters, Nagoya Math. J. 186 (2007) 173–191.
[3] S. Eidelman, Izdatelstvo Nauka, Moskva, 1964, (in Russian).
[4] Y. Hu, D. Nualart, Regularity of renormalized self-intersection local time for fractional brownian motion, A Journal of Communications in Information and Systems (CIS) 7 (2007) 21–30.
[5] P. Imkeller, V. Perez-Abreu, J. Vives, Chaos expansions of double intersection local time of brownian motion in r^d and renormalization, Stoch. Proc. Appl. 56 (1995) 1–34.
[6] N. Krylov, Some estimates of the probability density of a stochastic integral, Mathematics of the USSR-Izvestiya 8 (1974) 233–254.
[7] J.F. Le Gall, Wiener sausage and self-intersection local times, Journal of Functional Analysis 88 (1990) 299–341.
[8] P. Malliavin, Stochastic Analysis, Springer, 1997.
[9] D. Nualart, S. Ortiz-Latorre, Intersection local time for two independent fractional brownian motions, Journal of Theoretical Probability 20 (2007) 759–767.
[10] F.O. Porper, S.D. Eidel'man, Two-sided estimates of fundamental solutions of second-order parabolic equations, and some applications, Russian Mathematical Surveys 39 (1984) 119–178.
[11] J. Rosen, A renormalized local time for multiple intersections of planar brownian motion, Seminaire de Probabilities XX 20 (1986) 515–531.

[12] A. Rudenko, Local time for gaussian processes as an element of sobolev space, Commun. stoch. anal. 3 (2009) 223–247.
[13] B. Simon, The $P(\varphi)_2$ Euclidean (Quantum) Field Theory, Princeton University Press, 1974.
[14] A.Y. Veretennikov, N.V. Krylov, On explicit formulas for solutions of stochastic equations, Math. USSR-Sb. 29 (1976) 239–256.
[15] S. Watanabe, Lectures on Stochastic Differential Equations and Malliavin Calculus, Springer, 1984.
[16] A. Zvonkin, N. Krylov, About strong solutions of stochastic differential equations (in russian), in: Proceedings of the School-Seminar on the Theory of Random Processes, Druskininkai, November 25–30, 1974. Part II, Vilnius, 1975, pp. 9–88.